Permutations, Combinations and the Construction Method

A permutation of r objects from a set of n objects
refers to a situation in which
you are given a set of n objects, which you can distinguish from each other in some way,
and you need to choose r distinct objects from that set,
and you need to arrange those r objects in some order.
You may not choose any object more than once ("WITHOUT REPLACEMENT"),
any object can be placed in any position of an ordering,
and choosing the same r objects and arranging them in 2 different orderings
counts as 2 separate permutations ("ORDER MATTERS").

A combination of r objects from a set of n objects
refers to a situation in which
you are given a set of n objects, which you can distinguish from each other in some way,
and you need to choose r distinct objects from that set.
You DO NOT need to arrange those r objects in some order.
You may not choose any object more than once ("WITHOUT REPLACEMENT"),
and choosing the same r objects and arranging them in 2 different orderings
counts as only one combination ("ORDER DOESN'T MATTER").

Examples:

Your student ID is an 8 digit number, in other words 8 digits chosen from a set of 10 digits (0 to 9).
However, your student ID is not a permutation, because any digit can occur more than once,
and because the digit 0 cannot be placed in the first position of a student ID.
And your student ID is not a combination, because different orderings of the same digits count as different student IDs.

A board consisting of a president, a vice president and a treasurer,
all chosen from the 10 members of a group, and all positions being filled by different people,
can be viewed as 3 distinct people chosen from a set of 10 people.
The board is a permutation, because no person can be in 2 separate positions,
any person can fill any position,
and choosing one person to be the president, and a second person to be the vice-president,
is different than choosing the second person to be the president, and the first person to be the vice-president.

A board consisting of 3 members, all with the same title and responsibilities,
all chosen from the 10 members of a group, and all positions being filled by different people,
can be viewed as 3 distinct people chosen from a set of 10 people.
The board is a combination, not a permutation, because no person can be in 2 separate positions,
any person can fill any position,
and choosing person X to be the first person on the board, and person Y to be the next,
is the same as choosing person Y to be the first person on the board, and person X to be the next.

The number of permutations of r objects from a set of n objects is denoted by $P(n,r)$ or ${}_nP_r$ and equals $\dfrac{n!}{(n-r)!}$.

This is also called the number of permutations of n objects taken r at a time.

The number of combinations of r objects from a set of n objects is denoted by $C(n,r)$ or ${}_nC_r$ or $\dbinom{n}{r}$ and equals $\dfrac{n!}{(n-r)!r!}$.

This is also called the number of combinations of n objects taken r at a time.

Although you can use the formula for $P(n,r)$ to get the answer for quite a few permutation questions,
it is actually much better for your learning to solve them using the Construction Method,
because the Construction Method can be used to solve problems in which order matters,
but the other restrictions of a permutation do not necessarily have to be met.

To use the Construction Method to find out how many ways there are to make an arrangement:

1. Write a procedure for constructing an arrangement which satisfies the necessary restrictions.
   Often, your procedure addresses the strictest restriction first,
   and the loosest restriction last.
2. Verify that your procedure generates every arrangement that meets the necessary restrictions.
3. Verify that your procedure does not generate any arrangement that doesn't meet the necessary restrictions.
4. Verify that your procedure does not generate any arrangement that meets the necessary restrictions in more than 1 way.
5. Multiply the number of ways each step could be performed.
   NOTE: For each step, the number of ways it could be performed must be a single number.
   If there are a different number of ways to perform one step based on how a previous step was performed,
   the procedure may need to be broken into cases, or may need to be rethought completely.

Violating step 2 would give an answer that is too small (undercounting).
Violating steps 3 and 4 would give an answer that is too big (overcounting).

Example: How many 3 digit code numbers contain one digit that occurs twice, and one other (different) digit ?
808, 227 and 300 are examples of valid code numbers
444 is an example of an invalid code number

NOTE: This is neither a permutation nor combination problem (based on the description on page 1).
Since one digit will be repeated twice, it is not a permutation problem.
Since the ordering of the digits matters, it is not a combination problem.

Strategy: Try to construct a 3 digit code number that contains one digit that occurs twice, and one other (different) digit.

<u>Procedure 1:</u>
Step 1: Pick a number for the first digit.
Step 2: Use the same number for the second digit.
Step 3: Pick a different number for the third digit.
COMMENTS: The procedure can generate the valid number 227.
The procedure cannot generate the invalid number 444.
ERRORS: The procedure cannot generate the valid numbers 808 and 300.

<u>Procedure 2:</u>
Step 1: Pick a number for the first digit.
Step 2: Pick a number for the second digit.
Step 3: Use either the number from Step 1 or Step 2 for the third digit.
COMMENTS: The procedure can generate the valid numbers 808 and 300.
ERRORS: The procedure cannot generate the valid number 227.
The procedure can generate the invalid number 444.

<u>Procedure 3:</u>
Step 1: Pick a number for the first digit.
Step 2: Pick a different number for the second digit.
Step 3: Use either the number from Step 1 or Step 2 for the third digit.
COMMENTS: The procedure can generate the valid numbers 808 and 300.
The procedure cannot generate the invalid number 444.
ERRORS: The procedure cannot generate the valid number 227.

<u>Procedure 4:</u>
Step 1: Pick a number and use it for either the first, second or third digit.
Step 2: Use the same number for another digit.
Step 3: Pick a different number for the remaining digit.
COMMENTS: The procedure can generate the valid numbers 808, 227 and 300.
(In fact, the procedure can generate every valid number.)
The procedure cannot generate the invalid number 444.
(In fact, the procedure cannot generate an invalid number.)
ERRORS: The procedure can generate the valid number 227 in more than 1 way.
First way:
Step 1: pick 2 for the first digit
Step 2: use the 2 for the second digit
Step 3: pick 7 for the third digit
Second way:
Step 1: pick 2 for the second digit
Step 2: use the 2 for the first digit
Step 3: pick 7 for the third digit

Procedure 5: Step 1: Pick a number to be repeated for 2 digits.
Step 2: Use the number for all but one digit.
Step 3: Pick a different number for the remaining digit.

COMMENTS: The procedure can generate the valid numbers 808, 227 and 300.
(In fact, the procedure can generate every valid number.)
The procedure cannot generate the invalid number 444.
(In fact, the procedure cannot generate an invalid number.)
The procedure can generate the valid number 227 in only 1 way.
Step 1: pick 2 for the repeated digit
Step 2: use it for the first and second digits
Step 3: pick 7 for the third digit
(In fact, the procedure can generate every valid number in only 1 way.)
This procedure satisfies all the requirements for the Construction Method.

Now to answer the original question "How many 3 digit code numbers contain one digit that occurs twice, and one other (different) digit ?":
Step 1 can be done in 10 different ways (choose any number from 0 to 9 (ie. from a set of 10 numbers)).
Step 2 can be done in 3 different ways (choose either the first, second or third digit not to use the repeated number in (ie. from a set of 3 digits))
Step 3 can be done in 9 different ways (choose any number from 0 to 9 that wasn't chosen in Step 1 (ie. from a set of 9 numbers)).
The number of 3 digit code numbers that contain one digit that occurs twice, and one other (different) digit, is $10 \times 3 \times 9 = 270$.

NOTE: It can be quite tricky checking if an answer to a combinatorics question is correct.

One surefire (but potentially tedious) method is exhaustion (listing and counting all the possible arrangements).
However, that seems self-defeating,
since the point of solving these problems using these types of methods is to avoid the exhaustive work of finding and counting every possible arrangement.

One way to perform a quasi-check ("sanity check") is to use the method to solve the identical problem
with a much smaller set of objects to choose from,
and compare that answer against the count derived from the method of exhaustion.

For example, if you were restricted to using only the numbers 0 and 1,
the list of valid code numbers would be
100, 010, 001,
011, 101, 110
For a grand total of 6 code numbers.

Following Procedure 5 (but with only 2 numbers to choose from) gives $2 \times 3 \times 1 = 6$, the same answer.
So, we have more confidence in our approach for the larger case (with 10 numbers).

In order, to use this method of quasi-checking,
you must be very systematic in organizing the list of valid arrangements.

For example, if you were restricted to using only the numbers 0, 1 and 2,
the list of valid code numbers would be
100, 200, 010, 020, 001, 002,
011, 211, 101, 121, 110, 112,
022, 122, 202, 212, 220, 221
For a grand total of 18 code numbers.

(Following Procedure 5 (but with only 3 numbers) gives $3 \times 3 \times 2 = 18$, the same answer.)

The list of code numbers above was generated by
splitting them into 3 groups based on which number was repeated
(line 1 had repeated 0's, line 2 had repeated 1's, line 3 had repeated 2's),
then subgrouping each group according to which digit the non-repeated number was used in
(the first pair of each line had the non-repeated number in the first digit,
the second pair of each line had the non-repeated number in the second digit etc.),
then listing all possibilities for the non-repeated number in order (0, 1 then 2).

Now consider a variation:

Example: How many 3 digit numbers contain one digit that occurs twice, and one other (different) digit ?
(The difference is that a 3 digit number cannot start with a 0.)
808, 227 and 300 are examples of valid numbers
007, 011 and 444 are examples of invalid numbers

Comments: This variation involves more than just a minor change to our previous procedure,
because, even though the 0 cannot be the first digit,
it can appear either once or twice as the second and/or third digits.
This means that 0 could be selected during either Step 1 or Step 3 of our previous procedure.

SOLUTION 1: Split the original problem into three:

[a] How many 3 digit numbers contain one non-0 digit that occurs twice, and one 0 ?
[b] How many 3 digit numbers contain two 0's, and one non-0 digit ?
[c] How many 3 digit numbers contain one non-0 digit that occurs twice, and one other (different) non-0 digit ?

[a] How many 3 digit numbers contain one non-0 digit that occurs twice, and one 0 ? (eg. 808)

Procedure 6: Step 1: Pick a digit (other than the first) for the 0 to be used for.
Step 2: Pick a non-0 number to use for both remaining digits.
COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in 2 different ways (choose either the second or third digit to use the 0 in (ie. from a set of 2 digits)).
Step 2 can be done in 9 different ways (choose a non-0 number (ie. from a set of 9 numbers)).
The number of 3 digit numbers that contain one non-0 digit that occurs twice, and one 0, is 2×9.

[b] How many 3 digit numbers contain two 0's, and one non-0 digit ? (eg. 300)

Procedure 7: Step 1: Pick a non-0 number for the first digit.
Step 2: Use 0 for both remaining digits.
COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in 9 different ways (choose a non-0 number (ie. from a set of 9 numbers)).
Step 2 can be done in only 1 way.
The number of 3 digit numbers that contain two 0's, and one non-0 digit, is 9×1.

[c] How many 3 digit numbers contain one non-0 digit that occurs twice, and one other (different) non-0 digit ? (eg. 227)

Procedure 8: Step 1: Pick a non-0 number to be repeated for 2 digits.
Step 2: Use the number for all but one digit.
Step 3: Pick a different non-0 number for the remaining digit.
COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in 9 different ways (choose any number from 1 to 9 (ie. from a set of 9 numbers)).
Step 2 can be done in 3 different ways (choose either the first, second or third digit not to use the repeated number in (ie. from a set of 3 digits))
Step 3 can be done in 9 different ways (choose any number from 1 to 9 that wasn't chosen in Step 1 (ie. from a set of 8 numbers)).
The number of 3 digit numbers that contain one non-0 digit that occurs twice, and one other (different) non-0 digit, is $9 \times 3 \times 8$.

Combining the three solutions,

The number of 3 digit numbers that contain one digit that occurs twice, and one other (different) digit, is $2 \times 9 + 9 \times 1 + 9 \times 3 \times 8 = 243$.

SOLUTION 2: Find how many of our previously valid code numbers are now invalid, and subtract that from our previous count:
How many 3 digit code numbers contain one digit that occurs twice, and one other (different) digit, and start with a 0 ?
This problem must be split into two:

[a] How many 3 digit code numbers contain one non-0 digit that occurs twice, and one 0, and start with the 0 ?
[b] How many 3 digit code numbers contain two 0's, and one non-0 digit, and start with a 0 ?

[a] How many 3 digit code numbers contain one non-0 digit that occurs twice, and one 0, and start with the 0 ? (eg. 011)

Procedure 9: Step 1: Use 0 for the first digit.
Step 2: Pick a non-0 number to use for both remaining digits.

COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in only 1 way.
Step 2 can be done in 9 different ways (choose a non-0 number (ie. from a set of 9 numbers)).
The number of 3 digit numbers that contain one non-0 digit that occurs twice, and one 0, and start with the 0, is 1×9.

[b] How many 3 digit code numbers contain two 0's, and one non-0 digit, and start with a 0 ? (eg. 080 or 007)

Procedure 10: Step 1: Use 0 for the first digit.
Step 2: Use 0 for one other digit.
Step 3: Pick a non-0 number to use for the remaining digit.
COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in only 1 way.
Step 2 can be done in 2 different ways (choose either the second or third digit to use the 0 in (ie. from a set of 2 digits)).
Step 3 can be done in 9 different ways (choose a non-0 number (ie. from a set of 9 numbers)).
The number of 3 digit numbers that contain two 0's, and one non-0 digit, and start with a 0, is $1 \times 2 \times 9$.

Combining the two solutions, and subtracting from the original total,

The number of 3 digit numbers that contain one digit that occurs twice, and one other (different) digit, is $270 - 1 \times 9 - 1 \times 2 \times 9 = 243$.

NOTE: The two solutions give the same answer.

If the two answers were different, it would indicate that at least one of the two solutions is wrong.

Even if the two answers are the same, it is still possible that both solutions are wrong.
However, it would seem much less likely.

In fact, it is often a good idea to solve each combinatorics problem in more than 1 way if possible,
and verify that the answers are the same,
as it is very easy to either undercount or overcount.

Sometimes, after seeing the final answer for a combinatorics question, you can reverse engineer the answer to find an alternate solution.

For example, $243 = 9 \times 3 \times 9$, which suggests a possible solution involving
choosing a number (which can be done in 9 ways if 0 is excluded),
doing something else (which can be done in 3 ways – but what ?),
and then choosing another number (which can be done in 9 ways if the previously chosen number is excluded, but 0 is included).

SOLUTION 3:

Procedure 11: Step 1: Pick a non-0 number for the first digit.
Step 2: Decide if that number will be repeated in the second digit, repeated in the third digit, or not repeated at all.
Step 3: Pick a different number for the remaining digit(s).
COMMENTS: The procedure can generate every valid number in only 1 way, and cannot generate any invalid numbers.

Step 1 can be done in 9 different ways (choose a non-0 number (ie. from a set of 9 numbers)).
Step 2 can be done in 3 different ways.
Step 3 can be done in 9 different ways (choose any number from 0 to 9 that wasn't chosen in Step 1 (ie. from a set of 9 numbers))
The number of 3 digit numbers that contain one digit that occurs twice, and one other (different) digit, is $9 \times 3 \times 9 = 243$.

NOTE: If you were restricted to using only the numbers 0, 1 and 2,
the list of valid code numbers would be
100, 200,
211, 101, 121, 110, 112,
122, 202, 212, 220, 221
For a grand total of 12 numbers.

(Following Procedure 11 (but with only 3 numbers) gives $2 \times 3 \times 2 = 12$, the same answer.)